Simple Model for Strong Interactions*

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A simple model is proposed which relates elastic and inelastic processes in strong interactions. It is assumed that the basic process in strong interactions is meson production. The meson fields are calculated from a Klein-Gordon equation using a Gaussian source function. It is assumed that there are three types of strong interactions, whose quanta are, respectively, the π meson, the K meson, and the antibaryon. The sizes of the three corresponding source functions are inversely proportional to the square roots of the masses of the quanta. The three coupling constants are taken to be equal. Elastic scattering is assumed to be the diffraction scattering associated with the inelastic meson-producing processes. We thus obtain predictions for three phenomena in nucleon-nucleon collisions: the differential elastic-scattering cross section; the differential cross section for production of particles; the average number of mesons produced as a function of incident energy. The predictions are in good agreement with all experimental data in the multi-GeV region.

I. INTRODUCTION

THERE has recently been considerable interest in trying to understand something of the nature of the strong or nuclear interactions. A simple field theory of strong interaction is proposed, which explains several phenomena observed in scattering experiments. Elastic and inelastic processes are related to each other.

It is assumed that the meson fields obey an equation of the form

$$(\square+m^2)\phi=\rho. \qquad (1)$$

The quantity ρ is the source function for meson fields which arise whenever two nucleons interact. A functional form of ρ is chosen which leads to results that are in agreement with experiment. This source function is assumed to have three spatial regions, corresponding to three different types of strong interactions. The π meson, K meson, and antibaryon are the quanta of these three types of interactions. The three regions have progressively smaller radii corresponding to the increasing masses of the quanta.

The wave equation is solved employing the well-known in-out formalism. This leads to the result that the probability of producing n mesons is Poisson distributed. The expectation value of n, which parameterizes the Poisson distribution, contains the important information about scattering cross sections for elastic and inelastic processes. In fact we obtain predictions for several different phenomena. (1) The elastic differential cross section for nucleon-nucleon scattering. (2) The momentum-distribution function of the mesons produced in a high-energy nucleon-nucleon collision. (3) The multiplicity or total number of mesons produced as a function of energy. These predictions seem to be in rather good agreement with existing experimental data in the high-energy region. However, the data on inelastic processes are not very extensive and the validity of this model can only be tested by more experiments on inelastic processes at high energy.

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II. ELASTIC SCATTERING

There has recently been considerable interest in finding a simple explanation for the differential elastic-scattering cross sections that have been observed in strong interactions. It will be shown that all high-energy proton-proton elastic-scattering data can be explained in terms of the simple absorbing model proposed in an earlier paper.[1] It is especially interesting that the size of the absorbing region is rather energy-independent. This can be most easily seen by plotting X versus τ as was suggested.[1,2] This is done in[3] Fig. 1 where there is little energy dependence.

The normalized differential scattering cross section

$$X=(d\sigma/d\Omega_{\mathrm{c.m.}})(4\pi/k\sigma_{\mathrm{tot}})^2 \qquad (2)$$

has been plotted against the variable $-t$,[4] in the hope that this would result in an energy-independent curve—that is, one in which the diffraction peak does not shrink. There is in fact no reason why this should be so. An interaction region whose size is independent of energy does not necessarily result in an X plot which is independent of energy. As shown by Cocconi *et al.*[3] such plots have considerable energy dependence. A pure optical model encourages the hope that such a plot might be energy-independent. However, the pure optical model which worked so well for lower energy nucleon-nucleus scattering can not be expected to work in detail in the multi-GeV region, although it does successfully predict trends.[4]

[1] A. D. Krisch, Phys. Rev. Letters **11**, 217 (1963).

[2] It was independently pointed out that the transverse momentum may be a relevant variable in proton proton elastic scattering by D. S. Narayan and K. V. L. Sarma, Phys. Letters **5**, 365 (1963).

[3] G. Cocconi, V. T. Cocconi, A. D. Krisch, J. Orear, R. Rubinstein *et al.*, Phys. Rev. Letters **11**, 499 (1963); W. F. Baker, E. W. Jenkins, A. L. Read, G. Cocconi *et al.*, *ibid.* **12**, 132 (1964), A. N. Diddens, E. Lillethun, G. Manning, A. E. Taylor, T. G. Walker, and A. M. Wetherell, *Proceedings of the 1962 International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN Scientific Information Service, Geneva, 1963), p. 576. W. F. Baker, E. W. Jenkins, A. L. Read, G. Cocconi, V. T. Cocconi, and J. Orear, Phys. Rev. Letters **9**, 221 (1962). K. J. Foley, S. J. Lindenbaum, W. A. Love, S. Ozaki, J. J. Russell, and L. C. L. Yuan, *ibid.* **10**, 376 (1963); **11**, 425 (1963).

[4] R. Serber, Phys. Rev. Letters **10**, 357 (1963).

Thus, one is justified in studying functions of τ which are slightly different from X. It was found that by plotting the function

$$Y=Xe^{-4\sin^2\theta}=(d\sigma/d\Omega_{\mathrm{c.m.}})(4\pi/k\sigma_{\mathrm{tot}})^2e^{-4\sin^2\theta} \quad (3)$$

against $\tau=p^2\sin^2\theta$ essentially all the shrinkage was removed. This is shown in Fig. 2 where all experimental data[3,5] from 2.3 to 30 GeV have been plotted. The curve

$$Y=e^{-9.40\tau}+10^{-2.69}e^{-2.925\tau}+10^{-5.90}e^{-1.303\tau} \quad (4)$$

is also plotted. Thus, all proton-proton elastic scattering in the high-energy region can be fit by the simple sum of three exponentials in τ, the transverse momentum squared.

$$\frac{d\sigma}{d\Omega_{\mathrm{c.m.}}}=\left(\frac{k\sigma_{\mathrm{tot}}}{4\pi}\right)^2e^{+4\sin^2\theta}\left[e^{-9.40\tau}+10^{-2.69}e^{-2.925\tau}+10^{-5.90}e^{-1.303\tau}\right]. \quad (5)$$

It should be emphasized that no special significance is given to the term $e^{-4\sin^2\theta}$. It is merely a computational device used to get all experimental data on one line.

One can obtain a useful picture of the interaction by projecting out the partial-wave amplitudes from Eq. (4)

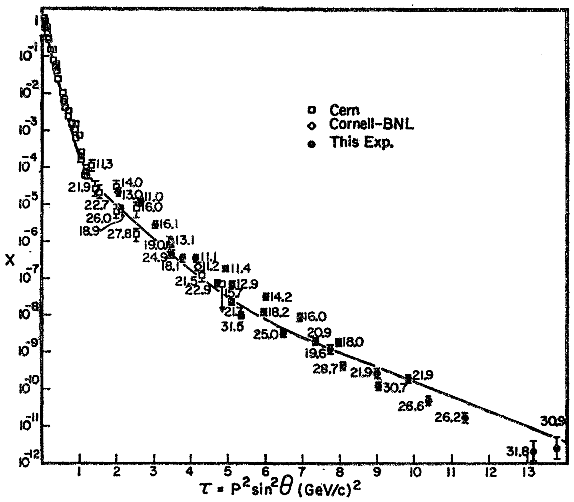

FIG. 1. The normalized differential scattering cross section $X=(d\sigma/d\Omega(4\pi/k\sigma_{\mathrm{tot}})^2$ is plotted against the transverse momentum squared $\tau=p^2\sin^2\theta$. All proton-proton elastic-scattering data above 10 GeV is shown. Some values of incident laboratory momenta are given.

according to the relation

$$1-b_l=\frac{k}{2}\int_{-1}^{1}dzP_l(z)(d\sigma/d\Omega_{\mathrm{c.m.}})^{1/2}, \quad (6)$$

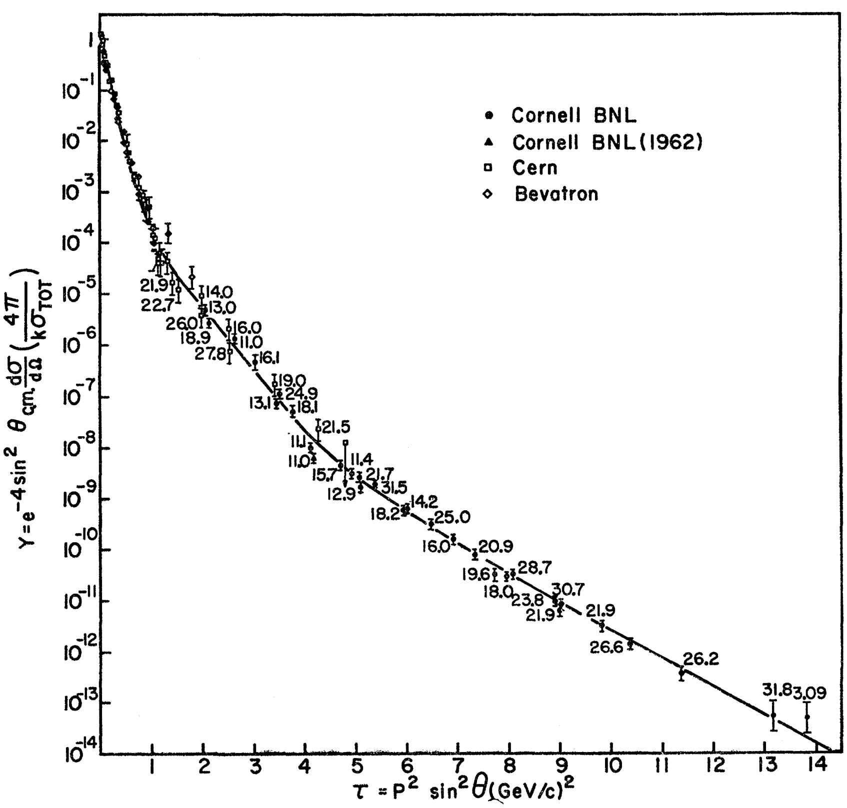

FIG. 2. The unconventionally normalized differential scattering cross section $Y=e^{-4\sin^2\theta}(d\sigma/d\Omega)\times(4\pi/k\sigma_{\mathrm{tot}})^2$ is plotted against the transverse momentum squared $\tau=p^2\sin^2\theta$. All proton-proton elastic-scattering data above 2.3 GeV are shown (Refs. 3, 5). Some values of the incident laboratory momenta are given. Equation (4) is also plotted.

[5] B. Cork, W. A. Wenzel, and C. W. Causey, Phys. Rev. 107, 859 (1957).

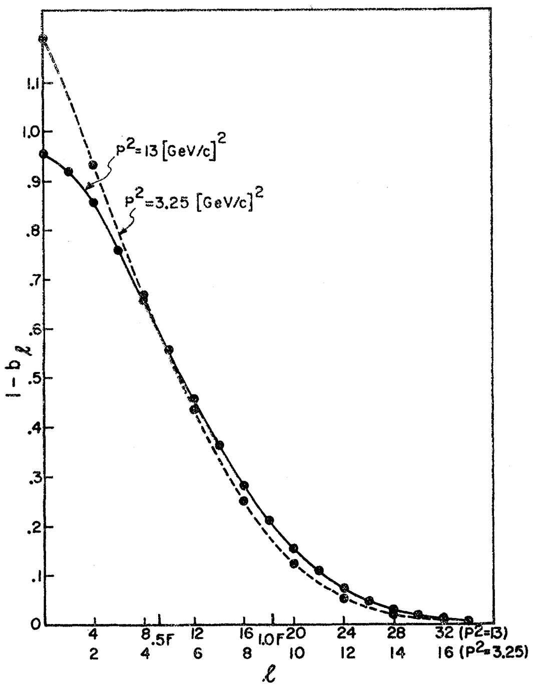

FIG. 3. The partial-wave amplitudes, $1-b_l$ are plotted against the "spatially" scaled values of the angular momentum, for center-of-mass momenta squared of $p^2=13$ $(\mathrm{GeV}/c)^2$ and $p^2=3.25$ $(\mathrm{GeV}/c)^2$.

where $b_l=e^{-\chi_l}$ is the transmission coefficient and χ_l is twice the imaginary part of the phase shift. The integrals involved are not elementary but they are similar to the $I_{ab}{}^l$ encountered previously[1] [in Eq. (11)], and can be done by computer. The projection is done for two different energies: $p_{\mathrm{c.m.}}{}^2=13$ $(\mathrm{GeV}/c)^2$ corresponding to $p_{\mathrm{lab}}=29$ GeV/c and $p_{\mathrm{c.m.}}{}^2=3.25$ GeV/c^2 corresponding to $p_{\mathrm{lab}}=7.3$ GeV/c. Both results are plotted in Fig. 3 on a scale in which the association $R=l/k$ has been made. Thus, each angular momentum is scaled down by the center-of-mass momentum so that we have equivalent "spatial" distributions. It is very striking that the resulting distribution is rather energy-independent.

χ_l, which is twice the imaginary part of the phase shift is tabulated in Table I and plotted in Fig. 4, where the same scaling procedure has been employed as in Fig. 3. χ_l has been interpreted[1] as the perpendicular interaction probability density. This is also seen to be rather independent of energy. In fact, the width of the distribution changes by less than 10% when $p_{\mathrm{c.m.}}{}^2$ changes from 13 to 3.25 $(\mathrm{GeV}/c)^2$.

III. MESON FIELD EQUATION

The above diffraction phenomenon can be explained in terms of a simple model for strong interactions. Assume that the basic process in strong interactions is meson production. This is analogous to the fact that photon production is the basic process of electromagnetic interactions. Then the strong elastic scattering can be understood as the diffraction scattering associated with these "inelastic" meson-production processes. Information about the nature of meson production can be obtained from the differential elastic-scattering cross section.[6]

We postulate that the meson can be described by a field $\phi(r)$. The field is assumed to obey a relativistic

TABLE I. $1-b_l$, the partial-wave amplitude and χ_l, which is twice the imaginary part of the phase shift, are tabulated for even integer angular momenta.

	$p^2=13$ $(\mathrm{GeV}/c)^2$		$p^2=3.25$ $(\mathrm{GeV}/c)^2$	
l	$1-b_l$	χ_l	$1-b_l$	χ_l
0	0.957	3.14	1.191	
2	0.920	2.52	0.932	2.69
4	0.856	1.94	0.668	1.10
6	0.760	1.43	0.436	0.573
8	0.659	1.075	0.250	0.288
10	0.558	0.816	0.123	0.131
12	0.456	0.609	0.052	0.053
14	0.364	0.452	0.019	0.019
16	0.281	0.330	0.007	0.007
18	0.211	0.237	0.003	0.003
20	0.153	0.166		
22	0.107	0.113		
24	0.072	0.075		
26	0.046	0.047		
28	0.029	0.030		
30	0.017	0.017		
32	0.010	0.010		
34	0.006	0.006		

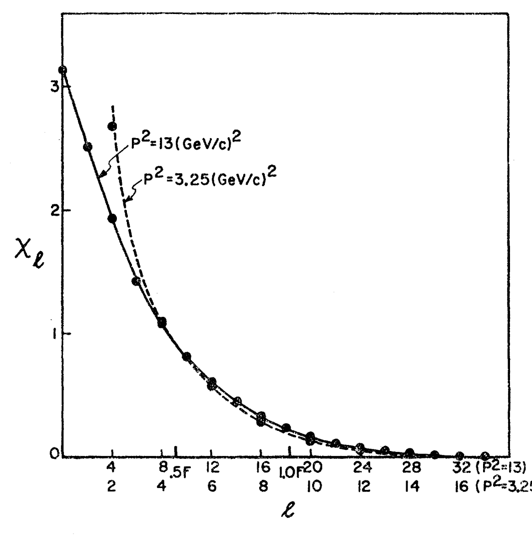

FIG. 4. χ_l, which is twice the imaginary part of the phase shift, or the perpendicular interaction probability density is plotted against the "spatially" scaled values of the angular momentum for $p^2=13$ $(\mathrm{GeV}/c)^2$ and $p^2=3.25$ $(\mathrm{GeV}/c)^2$.

[6] This idea is certainly not new. An interesting phenomenological discussion of the relation between elastic and inelastic scattering has recently been given by Van Hove. L. Van Hove, Nuovo Cimento 28, 798 (1963); Cargese Summer School, Corsica, 1963 (unpublished); CERN report 7963/TH. 392 (unpublished).

equation of the familiar form.[7]

$$(\Box+m^2)\phi(r)=\rho. \qquad (7)$$

The quantity ρ is the source function for the production of real mesons. It arises whenever two nucleons pass near each other. Thus ρ will depend on the distance between the two interacting nucleons, which we denote by $R=(\mathbf{R},T)$. It will also depend on the coordinate of the meson field with respect to the center of the source. The source is centered midway between the nucleons. This meson variable is denoted by $r=(\mathbf{r},t)$. For simplicity it is assumed that ρ is independent of all energies involved. Thus we have

$$\rho=\rho(r,R)\equiv g\bar{\rho}(r,R). \qquad (8)$$

The quantity g is the coupling constant of the strong interaction and $\bar{\rho}$ is normalized so that

$$\int \rho d^4r d^4R=\textit{finite}. \qquad (9)$$

All detailed information about the nature of strong interactions will then be contained in the functional dependence of ρ on r and R. At present, there is certainly no theory which gives information about the nature of the source function. Thus, the best approach seems to be to return to experiment and look for constraints on the form of ρ. The most important constraint comes from the well-known fact that the nuclear interaction is short range. This implies that ρ must fall off rapidly as a function of R, the internucleon distance. Due to the fact that at high energies the Lorentz contraction squashes everything down in the direction of the motion, it seems reasonable that in the parallel (Z) direction the dependence of ρ can be well represented by a δ function. It also seems reasonable to assume that ρ has a δ-function dependence on time. The actual interaction time is less than 10^{-24} sec. Other clues are contained in the fact that the differential elastic cross section and the differential meson-production cross sections seem to drop off as exponentials or Gaussians in the transverse momentum, $p_\perp=p\sin\theta$. All this evidence seems to point towards a source function which is a double Gaussian in the perpendicular direction.

$$\rho(r,R)=\frac{g}{(\pi a^2)^{1/2}}e^{-((2r_\perp)^2+R_\perp^2)/2a^2}\delta(Z)\delta(z)\delta(t). \qquad (10)$$

Of course, the validity of the above formula can only be tested in terms of the accuracy of the resulting predictions which will be developed in the following sections. The dependence on the meson field variable r cannot presently be too well tested because of a lack of good data on inelastic processes. Thus, it is possible that new data will indicate that some function other than the Gaussian is more reasonable. The Gaussian was chosen to obtain symmetry with respect to the internucleon variable R.

[7] This is just the Klein-Gordan equation with a source. It has been used by many people.

IV. SOLUTION OF WAVE EQUATION

The solution of Eq. (7) can be very nicely given in terms of the in-out formalism. The usefulness of this approach was first pointed out by Lewis, Oppenheimer, and Wouthuysen,[8] and has been pursued by others.[9] The situation is nicely reviewed in the book by Henley and Thirring[10] whose notation we will adopt. The meson wave function is given by

$$\phi^{\text{out}}=\phi^{\text{in}}+\int d^4r'\rho(r')\Delta(r'-r). \qquad (11)$$

The Δ function is the difference of the retarded and advanced Δ functions, and can be written exactly

$$\begin{aligned}\Delta(\mathbf{r},t)&\equiv\Delta^{\text{ret}}(\mathbf{r},t)-\Delta^{\text{adv}}(\mathbf{r},t)\\&=\mathbf{S}_k(e^{i\mathbf{k}\cdot\mathbf{r}}\sin\omega t/\omega), \quad -\infty<t<\infty. \end{aligned}\qquad (12)$$

Thus we have that

$$\begin{aligned}\phi^{\text{out}}-\phi^{\text{in}}&=\int d^4r\mathbf{S}_k\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\sin\omega(t-t')\rho(\mathbf{r}',t')}{\omega}\\&=-\mathbf{S}_k\frac{e^{i\mathbf{k}\cdot\mathbf{r}}\rho_k}{2i\omega}+\mathbf{S}_k\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}\rho_k^*}{2i\omega}.\end{aligned}\qquad (13)$$

The quantity ρ_k is the Fourier transform of the source function $\rho(r,R)$. It is given by

$$\rho_k=\int d^4r'\rho(r',R)e^{-ik\cdot r'}. \qquad (14)$$

It is now useful to give a Fourier expansion of ϕ^{in} and ϕ^{out} in terms of the in and out creation and annihilation operators. These are A_k, $A_k^\dagger$, B_k and $B_k^\dagger$. Thus we have

$$\begin{aligned}\phi^{\text{out}}&=\mathbf{S}_k\frac{B_ke^{i\mathbf{k}\cdot\mathbf{r}}}{[2\omega(2\pi)^3]^{1/2}}+\text{H.c.},\\ \phi^{\text{in}}&=\mathbf{S}_k\frac{A_ke^{i\mathbf{k}\cdot\mathbf{r}}}{[2\omega(2\pi)^3]^{1/2}}+\text{H.c.}\end{aligned}\qquad (15)$$

Putting these results into Eq. (13) we obtain the Fourier expansion of the solution

$$\begin{aligned}&\mathbf{S}_k\frac{B_ke^{i\mathbf{k}\cdot\mathbf{r}}}{[2\omega(2\pi)^3]^{1/2}}+\text{H.c.}\\&\quad=\mathbf{S}_k\frac{A_ke^{i\mathbf{k}\cdot\mathbf{r}}}{[2\omega(2\pi)^3]^{1/2}}+\text{H.c.}-\mathbf{S}_k\frac{\rho_ke^{i\mathbf{k}\cdot\mathbf{r}}}{2i\omega(2\pi)^3}+\text{H.c.}\end{aligned}\qquad (16)$$

[8] H. W. Lewis, J. R. Oppenheimer, and S. Wouthuysen, Phys. Rev. **73**, 127 (1948).

[9] E. M. Henley and T. D. Lee, Phys. Rev. **101**, 1536 (1955), Z. Koba and G. Takeda, Progr. Theoret. Phys. (Kyoto) **19**, 269 (1958).

[10] E. M. Henley and W. Thirring, *Elementary Quantum Field Theory* (McGraw-Hill Book Company, Inc., New York, 1962), Chap. 8, 9, 10.

Thus, we have that the in and out annihilation operators are related by the simple equation

$$B_k = A_k + i\rho_k/[2\omega(2\pi)^3]^{1/2}. \tag{17}$$

Now consider the state consisting of the two incoming nucleons and no real mesons with momentum k. This we denote by $|\text{in}, n_k=0\rangle$. We can get a matrix equation for $|\text{in}, n_k=0\rangle$ by considering $B_k|\text{in}, n_k=0\rangle$ and noting that $A_k|\text{in}, n_k=0\rangle=0$. We then have that

$$B_k|\text{in}, n_k=0\rangle = \frac{i\rho_k}{[2\omega(2\pi)^3]^{1/2}}|\text{in}, n_k=0\rangle. \tag{18}$$

Then we are in a position to evaluate the probability of producing n mesons with momentum k. This probability is given by

$$\begin{aligned} P(n_k) &= |\langle \text{out}, n_k|\text{in}, n_k=0\rangle|^2 \\ &= |\langle \text{out}, n_k=0|(B_k^{n_k}/(n_k!)^{1/2})|\text{in}, n_k=0\rangle|^2 \\ &= \left|\langle \text{out}, n_k=0|\left[\frac{i\rho_k}{[2\omega(2\pi)^3]^{1/2}}\right]^{n_k}\frac{1}{(n_k!)^{1/2}}|\text{in}, n_k=0\rangle\right|^2 \\ &= \frac{1}{n_k!}\left[\frac{\rho_k^2}{2\omega(2\pi)^3}\right]^{n_k}|\langle \text{out}, n_k=0|\text{in}. n_k=0\rangle|^2. \end{aligned} \tag{19}$$

Now it can be shown that the normalization is such that

$$|\langle \text{out}, n_k=0|\text{in}, n_k=0\rangle|^2 = e^{-\bar{n}_k}. \tag{20}$$

The quantity $\bar{n}_k$ is the average number of mesons given by

$$\bar{n}_k = |\rho_k|^2/2\omega(2\pi)^3. \tag{21}$$

Thus the probability of producing n_k mesons in a state k is Poisson distributed according to

$$P(n_k) = (1/n_k!)\bar{n}_k^{n_k}e^{-\bar{n}_k}. \tag{22}$$

This Poisson distribution is parameterized by $\bar{n}_k$ which is the expectation value of the number of mesons. For a more detailed discussion of the above seen Henley and Thirring.[10]

We can now find the average probability densities for various processes by finding ρ_k, the Fourier transform of the source function and putting it into the above equation for $\bar{n}_k$. Using the double Gaussian source function proposed in Sec. III we have

$$\begin{aligned} \rho_k &= \int d^4r\rho(r,R)e^{-ik\cdot r} \\ &= \frac{2g}{(\pi a^2)^{1/2}}e^{-R_\perp^2/2a^2}\delta(Z)\int d^3r \\ &\quad \times e^{-i\mathbf{k}\cdot\mathbf{r}}e^{-2r_\perp^2/a^2}\delta(z)\int dt\delta(t)e^{i\omega t} \\ &= \frac{g}{(\pi a^2)^{1/2}}e^{-R_\perp^2/2a^2}\delta(Z)[\tfrac{1}{2}\pi a^2e^{-k_\perp^2a^2/8}][1], \\ &= g(\pi a^2)^{1/2}e^{-R_\perp^2/2a^2}e^{-k_\perp^2a^2/8}\delta(Z). \end{aligned} \tag{23}$$

Integrating over the nucleon-nucleon coordinate parallel to the initial motion (Z) we get the dependence of ρ_k on $R_\perp$ and $k_\perp$.

$$\rho_k(R_\perp,k_\perp) = ga\pi^{1/2}e^{-R_\perp^2/2a^2}e^{-k_\perp^2a^2/8}. \tag{24}$$

Then the expectation value for the number of mesons produced in a state characterized by $R_\perp$, $k_\perp$, and k_Z is given by

$$\bar{n}_k(R_\perp,k_\perp,k_Z) = \frac{|\rho_k|^2}{2\omega(2\pi)^3} = \left(\frac{ga}{4\pi}\right)^2\frac{1}{\omega}e^{-R_\perp^2/a^2}e^{-k_\perp^2a^2/4}. \tag{25}$$

We can remove the dependence on k_Z, the momentum parallel to the incident motion by integrating over this variable

$$\begin{aligned} \bar{n}(R_\perp,k_\perp) &= \int_{-p}^{p} dk_Z\bar{n}(R_\perp,k_\perp,k_Z) \\ &= \left(\frac{ga}{4\pi}\right)^2 e^{-R_\perp^2/a^2}e^{-k_\perp^2a^2/4}\int_{-p}^{p}\frac{dk_Z}{(k_Z^2+k_\perp^2+m^2)^{1/2}} \\ &= \left(\frac{ga}{4\pi}\right)^2 e^{-R_\perp^2/a^2}e^{-k_\perp^2a^2/4}\ln\frac{(p^2+m^2+k_\perp^2)^{1/2}+p}{(p^2+m^2+k_\perp^2)^{1/2}-p}. \end{aligned} \tag{26}$$

This expression is exact but unwieldy. In the high-energy limit $p \gg m$, $p \gg k_\perp$, it can be approximated by the expression

$$\bar{n}(R_\perp,k_\perp,p) = \left(\frac{ga}{4\pi}\right)^2 e^{-R_\perp^2/a^2}e^{-k_\perp^2a^2/4}\ln\frac{4p^2}{k_\perp^2+m^2}. \tag{27}$$

It is seen that this density has a logarithmic dependence on p^2, the center-of-mass momentum of the incident nucleons.

A very interesting quantity which can now be computed is the average number of mesons with transverse momentum $k_\perp$ that are produced when two nucleons collide with momentum p. This is obtained by integrating $\bar{n}(R_\perp,k_\perp,p)$ over the variable $R_\perp$

$$\begin{aligned} \bar{n}(k_\perp,p) &= \int d^2R_\perp\bar{n}(R_\perp,k_\perp,p) \\ &= \left(\frac{ga}{4\pi}\right)^2 e^{-k_\perp^2a^2/4}\ln\frac{4p^2}{k_\perp^2+m^2}2\pi\int_0^\infty e^{-R_\perp^2/a^2}R_\perp dR_\perp \\ &= \frac{g^2a^4}{16\pi}e^{-k_\perp^2a^2/4}\ln\frac{4p^2}{k_\perp^2+m^2}. \end{aligned} \tag{28}$$

We can also compute the average number of mesons produced at an impact parameter $R_\perp$. This function determines the nature of the elastic diffraction scattering.

$$\begin{aligned} \bar{n}(R_\perp,p) &= \int d^2k_\perp\bar{n}(R_\perp,k_\perp,p) \\ &= \left(\frac{ga}{4\pi}\right)^2 e^{-R_\perp^2/a^2}2\pi\left[\int_0^p e^{-k_\perp^2a^2/4}k_\perp dk_\perp \ln 4p^2 \right. \\ &\quad \left. - \int_0^p e^{-k_\perp^2a^2/4}k_\perp dk_\perp \ln(k_\perp^2+m^2)\right]. \end{aligned} \tag{29}$$

The first integral can be easily solved.

$$\int_0^p e^{-k_\perp^2 a^2/4} k_\perp dk_\perp = \frac{2}{a^2}(1-e^{-p^2a^2/4}) \approx \frac{2}{a^2}. \quad (30)$$

However, the second integral can be solved only in terms of an infinite series. Fortunately it will be essentially independent of p and thus can be written as a constant. Thus we have

$$\bar{n}(R_\perp,p) = (g^2/4\pi)e^{-R_\perp^2/a^2}[\ln 4p^2 - \ln c]. \quad (31)$$

Now we can interpret this constant in terms of the threshold energy for meson production by simply requiring that $\bar{n}(R_\perp,p_0)=0$. Then we have

$$\bar{n}(R_\perp,p) = (g^2/4\pi)e^{-R_\perp^2/a^2}\ln p^2/p_0^2. \quad (32)$$

Finally we can obtain an expression for the multiplicity or the total number of mesons produced in a nucleon-nucleon collision with momentum p.

$$\bar{n}(p) = \int d^2R_\perp \bar{n}(R_\perp,p)$$
$$= \frac{g^2}{4\pi}\ln\frac{p^2}{p_0^2} 2\pi \int_0^\infty R_\perp dR_\perp e^{-R_\perp^2/a^2} \quad (33)$$
$$= (\tfrac{1}{2}ga)^2 \ln(p^2/p_0^2).$$

To obtain useful experimental predictions we employ the fact that for nucleon-nucleon collisions $2p^2=MT$. The quantity T is the laboratory kinetic energy. Thus we have that

$$p^2/p_0 = T/T_0. \quad (34)$$

We then obtain the following density functions for the average number of mesons produced

$$\bar{n}(T) = (\tfrac{1}{2}ga)^2 \ln(T/T_0), \quad (35)$$

$$\bar{n}(R_\perp,T) = (g^2/4\pi)\ln(T/T_0)e^{-R_\perp^2/a^2}, \quad (36)$$

$$\bar{n}(k_\perp,T) = (g^2a^4/16\pi)\ln[2MT/(k_\perp^2+m^2)]e^{-k_\perp^2a^2/4}. \quad (37)$$

V. RESULTS

We now postulate that there are three types of strong interactions These may be characterized by their quanta, which are, respectively, the π meson, the K meson, and the antibaryon. The last two are characterized by the fact that they carry two important quantum numbers, strangeness and baryon number. The coupling constant g is assumed to be the same for all three of these interactions and all three are assumed to have double Gaussian source functions of the type given in Eq. (10). However, the radius of the source function a_i is assumed to depend on the mass of the quantum. In fact, it is assumed that

$$a_i = J/m_i, \quad (38)$$

where m_i is the mass of the quantum. To give the best fit to existing experimental data, J is taken to be 0.0945 GeV-F². The validity of this approach can be best seen by considering the quantity $\alpha = a^2/2(\hbar c)^2$. This is the slope in a plot of differential cross section versus τ. See Fig. 1. This slope is related to the spatial size of the interaction region by taking the Fourier transformation of the Gaussian in $p_\perp = \sqrt{\tau}$. We then calculate and tabulate the relevant quantities in Table II. The resulting slopes and interaction radii are in fairly good agreement with the experimental results for elastic proton-proton scattering. We have included in Table II a list of the threshold energies for the production of each meson in nucleon-nucleon collisions.

TABLE II. The various parameters that can be calculated for the three types of interactions.

Meson	Mass GeV	Interaction radius a F	Slope in τ plot α GeV/c^{-2}	Threshold energy T_0 GeV
π	0.140	0.82	8.73	0.29
K	0.495	0.44	2.47	1.75
$\bar{N}$	0.938	0.32	1.30	5.64

Finally we choose as our coupling constant $g^2 = 5.6$ F^{-2} for all three types of interactions. This is chosen to agree with some experimental results which will be described below. Thus we have only two arbitrary parameters in our theory, the coupling constant g and the interaction size parameter J.

We will now show how the expectation valve density $\bar{n}(R_\perp,T)$ is related to the elastic-scattering cross section. It has been shown in Sec. IV that the number of mesons

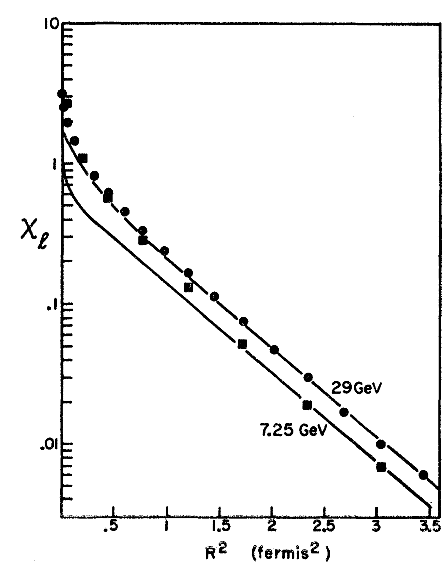

FIG. 5. Logχ_l, which is twice the imaginary part of the phase shift is plotted against $R_\perp^2$ in F² for incident laboratory momenta of 29 GeV and 7.25 GeV. The theoretical curves are calculated from Eq. (45). The points are the experimental values from elastic scattering given in Table I.

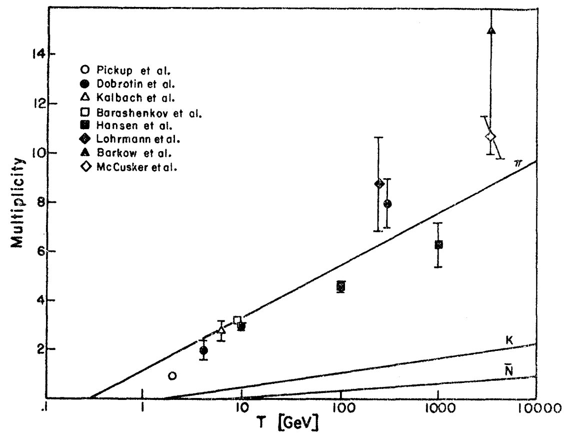

FIG. 6. The multiplicity of meson production in nucleon-nucleon collisions is plotted as a function of laboratory kinetic energy. Experimental points (Ref. 11) for π-meson production are plotted. Theoretical curves for π, K, and $\bar{N}$ production are calculated from Eq. (46).

produced is Poisson distributed. Thus the probability of producing no π mesons is

$$P_\pi(0)=e^{-\bar{n}_\pi(R_\perp,T)}. \tag{39}$$

Similarly the probabilities of producing no K's and $\bar{N}$'s are given by

$$P_K(0)=e^{-\bar{n}_K(R_\perp,T)}, \qquad P_{\bar{N}}(0)=e^{-\bar{n}_N(R_\perp,T)}. \tag{40}$$

Thus the probability of producing no mesons at all is given by the product of the three probabilities.

$$\begin{aligned}P_{\pi K\bar{N}}(0)&=P_\pi(0)P_K(0)P_{\bar{N}}(0)\\&=\exp-[\bar{n}_\pi(R_\perp,T)+\bar{n}_K(R_\perp,T)+\bar{n}_{\bar{N}}(R_\perp,T)].\end{aligned} \tag{41}$$

The probability of producing one or more particles of any type is simply one minus the probability of producing no mesons

$$\begin{aligned}P(\text{any number of particles})&=1-P_{\pi K\bar{N}}(0)\\&=1-e^{-[\bar{n}_\pi+\bar{n}_K+\bar{n}_N]}.\end{aligned} \tag{42}$$

But this is just the probability of having an inelastic interaction at impact parameter $R_\perp$. Now recall that the partial-waves expansion for the inelastic cross section is

$$\sigma_{IN}=(\pi/k^2)\sum_{l=0}^{\infty}(2l+1)(1-e^{-2\chi_l}). \tag{43}$$

The quantity $1-e^{-2\chi_l}$ is also the probability of having an inelastic interaction at $R_\perp$. Thus we have a simple relation between the imaginary part of the phase shift and the expectation value densities.

$$\chi_{l=kR_\perp}=\tfrac{1}{2}[\bar{n}_\pi(R_\perp,T)+\bar{n}_K(R_\perp,T)+\bar{n}_{\bar{N}}(R_\perp,T)]. \tag{44}$$

This is the χ_l which was calculated from the elastic-scattering experiments in Sec. II. These results are plotted in Fig. 5 on semilog paper to display the Gaussian nature of χ_l. A good fit is obtained using our choice of $g^2=5.6\,\mathrm{F}^{-2}$ for all three types of interactions. However, it is possible that a slightly better fit could be obtained by choosing different coupling constants for the K and $\bar{N}$ interactions. Using $g^2=5.6$, we obtain from Eq. (36)

$$\begin{aligned}\chi_{l=kR_\perp}=0.50[&\log(T/0.29)e^{-(R_\perp/0.82)^2}\\&+\log(T/1.75)e^{-(R_\perp/0.44)^2}\\&+\log(T/5.64)e^{-(R_\perp/0.32)^2}].\end{aligned} \tag{45}$$

This curve is also plotted in Fig. 5 for $T=29$ GeV and $T=7.25$ GeV. The agreement is rather good for $T=29$ GeV. For $T=7.25$ GeV, things begin to break down near the center. There are two reasons why this might be so: (1) 7.25 GeV is near threshold for antiparticle production. (2) For 7.25 GeV, the classical approximation involved in setting $l=kR_\perp$ is poor near the center. This is because the wavelength corresponding to a particle of this energy is no longer small with respect to $R_\perp$.

Using the same coupling constant we can now predict the multiplicity of the particles produced in nucleon nucleon collisions from Eq. (35). We have

$$\begin{aligned}\bar{n}_\pi(T)&=2.14\log(T/0.29),\\\bar{n}_K(T)&=0.62\log(T/1.75),\\\bar{n}_{\bar{N}}(T)&=0.32\log(T/5.64).\end{aligned} \tag{46}$$

These curves are plotted on semilog paper in Fig. 6. The experimental results[11] for π-meson multiplicity are also plotted. These include cosmic-ray results up to 3500 GeV. The fit is rather good. There are few experimental results on strange particle and antiparticle production. However, Eq. (46) is consistent with what results there are.[12] Observing high-energy protons interacting in a hydrogen bubble chamber would be a powerful test of this theory.

There are also predictions about the momentum distributions of the secondaries produced. These can be obtained from Eq. (37) by putting in appropriate

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[12] R. I. Louttit, T. W. Morris, D. C. Rahm, R. R. Rau, A. M. Thorndike, W. J. Willis, and R. M. Lea, Phys. Rev. **123**, 1465 (1961) studied strange particle production at 2.85 GeV.

factors of $\hbar c = 0.197$ GeV-F.

$$\bar{n}_\pi(p_\perp,T) = 2.87 \log\left(\frac{1.88T}{p_\perp^2+(0.140)^2}\right)e^{-4.30p_\perp^2},$$
$$\bar{n}_K(p_\perp,T) = 0.24 \log\left(\frac{1.88T}{p_\perp^2+(0.495)^2}\right)e^{-1.23p_\perp^2}, \quad (47)$$
$$\bar{n}_{\bar{N}}(p_\perp,T) = 0.065 \log\left(\frac{1.88T}{p_\perp^2+(0.938)^2}\right)e^{-0.65p_\perp^2}.$$

There have been few experiments which can be used to directly test these predictions. However, they do seem to be in vague agreement with the beam survey data.[13] Cocconi[14] has previously pointed out that these data could be fit by an exponential in $p_\perp$ while we propose a Gaussian in $p_\perp$. It should be noted that the exact validity of these predictions is to some extent independent of the rest of the theory. For this directly involves the dependence of the source function ρ (R,r) on the meson field variable, r. This dependence could be modified without changing the rest of the theory. Nevertheless the present simple double Gaussian is not inconsistent with existing experiments. There is no reason to make it more complex unless more accurate experiments show that it is not correct in detail.

The inelastic differential cross section for the production of charged particles in p-p collisions at 19.0 GeV was recently measured at CERN.[15] We have used these experimental results to calculate the total number of particles produced by using the equations

$$\begin{aligned}\pi &= \tfrac{3}{2}(\pi^+ + \pi^-),\\ K &= 2(K^+ + K^-),\\ \bar{N} &= 2\bar{p}.\end{aligned} \quad (48)$$

Thus, the uncharged particles are included. The resulting data points are plotted in Fig. 7. We also plot Eq. (47) using the following normalization

$$\frac{d^2\sigma}{dp\,d\Omega} = \sigma_{INEL}\bar{n}(p_\perp,T) = 18.5 \text{ mb } \bar{n}(p_\perp,T). \quad (49)$$

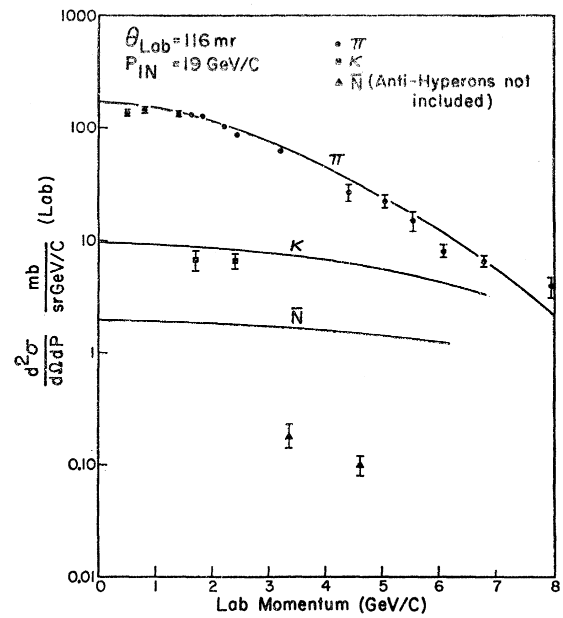

FIG. 7. Cross section for the production of π, K and $\bar{N}$ in 19.0 GeV/c p-p collisions. Equation (47) is also plotted.

The valve of $\sigma_{INEL} = 18.5$ mb is not equal to the experimental value of $\sigma_{INEL} = 30$ mb. Nevertheless, the agreement for π and K is quite good. Note that the $\bar{N}$ data points are below the theoretical curve. This is probably because the strange antibaryons were not observed in this experiment.

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